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TEACHING MATERIAL ON

MATHEMATICS

SCHOOL OF SCIENCE

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Optimization Method (53)

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Problem involving continuous and differentiable functions are analytical methods & make use of calculus to find points of maxima or minima.

(A) Linear Optimization Method (Linear O.F. & Linear constraints in decision variables) → Unconstrained Multivariable Opt. problem

(B) Non-Linear Optimization Method (O.F. — non linear or one or more constraints inequations have non-linear relationship or both; i.e., O.F. &/and the constraints are non-linear)

Unconstrained single variable optimization problem
N & S condition for local maxima & minimum value.

Unconstrained Multivariable optimization problem (i) Taylor's series expansion and (ii) using $H(x)$ Hessian Matrix

With Equality constraint

(i) ~~Direct~~ substitution Method
(ii) Lagranges Multipliers Method (for three variables)
(iii) Lagranges Multipliers Method for n-variables with m-equality constraints ($m \leq n$)
for Known concavity and convexity
(iv) LM-Method for unknown concavity and convexity.

With inequality constraints

(a) Khun Tucker Necessary conditions.
(b) Khun Tucker Sufficient conditions.

End

(B1) NON-Linear optimization Method

Unconstrained

B1

(i) By Lagrangian Function Model formulation with inequality constraints
(ii) GNLP method (General Non-linear Prog Method)
(iii) Quadratic Programming & its Applications
(a) Khun Tucker Conditions (Non-linear obj fn and linear constraints)
(b) Wolfe's Modified Simplex Method
(c) Beale's Method

(B2) STOCHASTIC PROGRAMMING PROB

(B3) Geometric Programming
Optimization problem involving special types of fns called Posynomials $\sum c_i \prod (x_i)^{a_{ij}}$ (real expression of the form) ($c_i > 0$)

(iii) Graphical Soln Method (B4)

Separable Programming

Indirect Programming Methods

(V) (a) Separable programming NLP problem in which obj fn & constraints eqns are separable
(b) Reducible to separable Programming
(c) Separable convex programming Problem
(d) Piecewise linear approximation of Non-linear Functions.
(e) Mixed-integer Approximation of separable NLP Problem.

* Separable convex programming problem
* Mixed integer Approximation of separable NLP problem

** Geometric programming Problem (54)

(a) General Mathematical programming
(b) Primal G. P. problem with equality constraints

** STOCHASTIC Programming Problem [Programming Problem in which parameters are unknown]

(i) Sequential stochastic programming
(ii) Non-sequential stochastic programming
(iii) Chance - constrained programming

(B5) Direct Programming Method

(i) Direct search Method
(ii) Descent / Gradient Method.

(B6) (There is another chapter) (Fractional Programming Problem in Non-linear Programming problems - specially used in Finance sector.

CLASSIFICATION OF OPTIMIZATION PROBLEMS

Problems involving continuous and differentiable functions which are analytical in nature and make use of calculus to find points of maxima or minima.

(A) Linear optimization Problem

(Those programming problems involving linear objective function and linear constraints in decision variables).

Constrained Multivariable optimization problems

Unconstrained single variable optimization problems

(Necessary & sufficient conditions for Local Maximum or Minimum value.)

Unconstrained Multivariable optimization problems.

Solutions using:
1) Taylor's series expansions
II) Hessian Matrix ($H(x)$) Method

With Equality Constraints

Solutions using:

(I) Direct Substitution Method
(II) Lagranges Multiplier Method (for three variables)
(III) Lagranges Multiplier Method (for n-variables and m-equality constraints ($m \le n$) for known concavity and convexity)
(IV) Lagranges Multiplier Method (for unknown concavity or convexity)

With inequality Constraints

Solutions using:

1) Khun Tucker Necessary conditions.
(II) Khun Tucker-Sufficient conditions.

(B) Non-linear optimization problem

(Those programming problems which have its non-linear objective function, or one or more constraints inequations having non-linear relationship or both, i.e., obj. function and/or the constraints are non-linear.)

I) Solution ~~by~~ unconstrained Method

(B1) By Lagrangian Function

* Modulel formulation using inequality Constraints
* GNLP method (General Non-Linear Programming Method)
* Graphical sol^n Method

(B2) Quadratic Programming and its application:—

* Khun Tucker Conditions (Non-linear objective & linear constraints)
* Wolfe's Modified Simplex Method
* Beale's Method

(B3) Geometric Programming Problems

Def: Optimization problem involving special types of functions called Posynomials

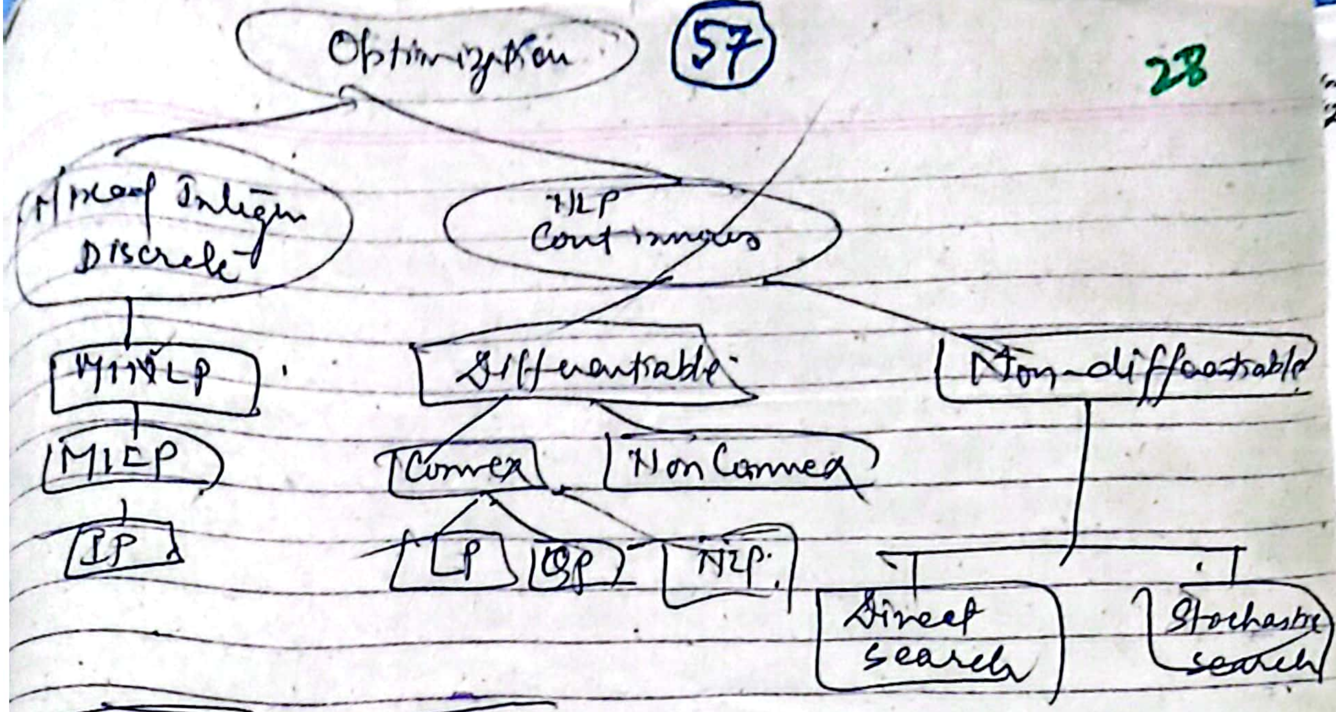
* General Mathematical Programming Problem
* Primal G.P. problem with equality constraints

(B4) Separable Programming Problems: (Indirect Prog. Problem)

* Separable NLP problem in which objective functions & constraints equations are separable.
* Reducible to separable Programming Problem
* Piece-wise linear approximation of Non-linear Functions
* Separable convex Programming problem
* Mixed integer approximation of separable NLP problem.

II)

III) STOCHASTIC Programming Problems.



Application of optimization in Engineering.

* Helps engineers in their design and analysis of systems, but also leads to significant advances and new discoveries in optimization theory and techniques.

Mechanics:—

* Problems in rigid body dynamics. to view rigid body dynamics as attempting to solve ordinary differential eqs. on a contraint manifold where constraints are various non-linear geometric constraints such as "these two points must always coincide; this surface must not penetrate any other; or this point must always lie somewhere on this curve". Also the problem of computing contact forces can be done by solving a linear complementary problem which can also be viewed as a QP problem.

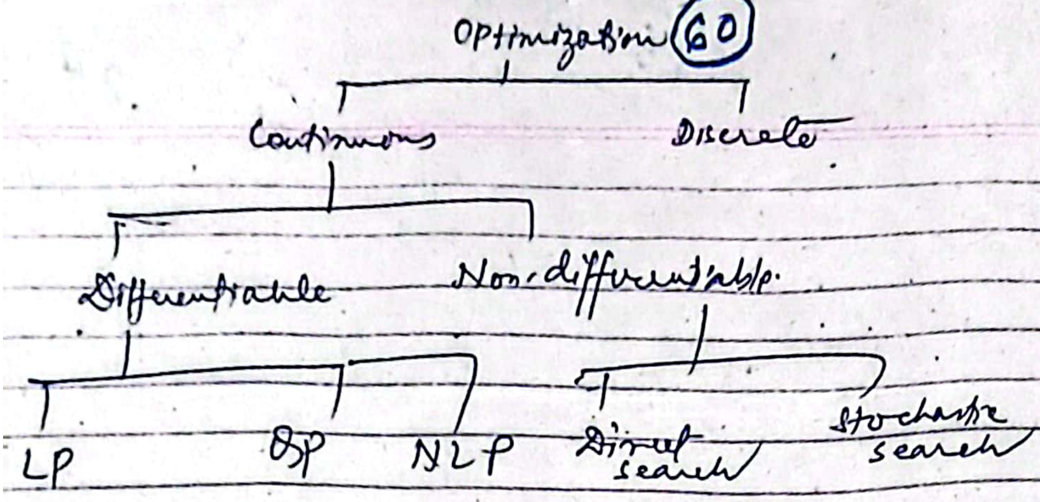
Many design problems can be expressed as optimization programs. The application is called design optimization. One subset of the engineering optimization and another recent and growing subset of this field is multidisciplinary design optimization which while useful in many problems, has in particular been applied to aerospace engineering problem.

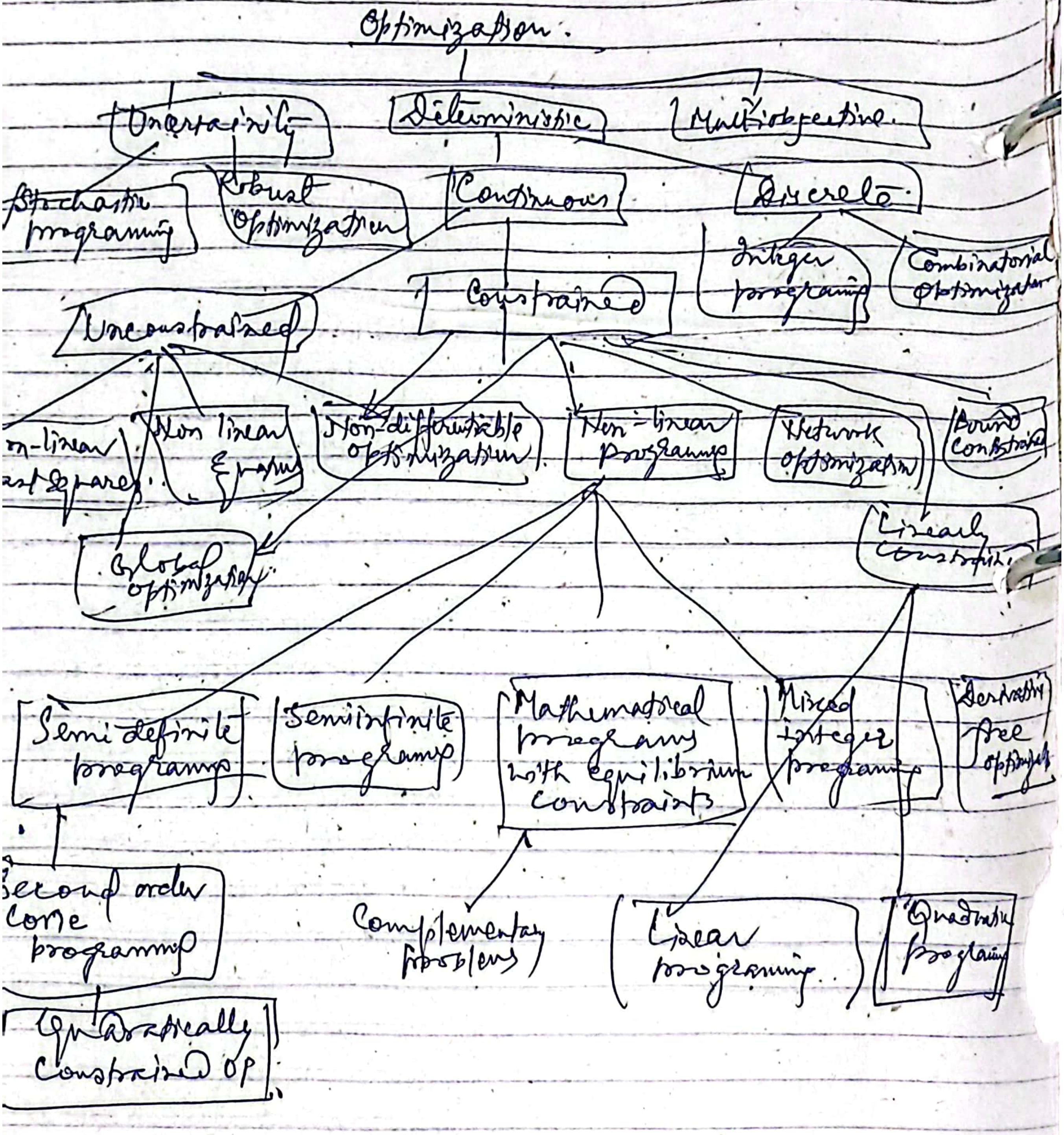
This approach may be applied in cosmology and astrophysics.

~~Economics & Finance~~ In Electrical Engineering :-

Optimization techniques in electrical engineering include active filter design, stray field reduction in super conductivity

Civil Engineering.


Optimization (60)
Continuous
Discrete
Differentiable
Non-differentiable
LP
QP
NLP
Direct search
Stochastic search


Optimization.
Uncertainity
Deterministic
Multiobjective.
Stochastic programming
Robust Optimization
Continuous
Discrete
Integer programing
Combinatorial optimization
Constrained
Unconstrained
Non-linear least squares
Non linear Eqns
Non-differentiable optimization
Non-linear programming
Network optimization
Bound Constrained
Global optimization
Linearly Constrained
Semi definite programming
Semiinfinite programming
Mathematical programs with equilibrium constraints
Mixed integer programming
Derivative free optimization
Second order cone programming
Complementary problems
Linear programming
Quadratic programing
Quadratically Constrained OP

Example search - optimization technique - historical background

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① Early classic optimization approaches :—

Formal optimization approach on "linear programming" was initiated by Leonid Kantorovich in 1939. ~~The first well~~

The Simplex Method was published in 1947 by George Dantzig, and the theory of duality was published contemporarily (in the same year almost) by John Von Neumann.

What is optimization?

Optimization is a word that tends to get misunderstood as each person seems to have their own interpretation of what optimization means.

- Engineers and technicians instinctively think of optimization as "trial and error"
- Others believe that optimization process is exhaustively listing the design possibilities and picking the best one.
- Then there are those who perceive that optimization is simply making qualitative suggestions that leads to a better product design.

However, are these methods of optimization? They are optimization methods in a broad sense but they are not proper ones if you follow the definition of optimization as a quantitative and systematic methodology to search for the best design among numerous possibilities while satisfying

$\Delta_3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$

given constraints. (62)

In language of mathematics, we can say that optimization is a process of finding out the conditions that give maximum or minimum values of a function and satisfying the existing constraints.

Optimization can be taken as minimization, since the maximum of a function can be obtained by finding the minimum of negative of the same function.

The optimization methods are also known as mathematical programming techniques. These techniques are used to find out the minimum of a fn of several variables under a prescribed set of constraints.

& **Statement of Optimization Problem:—**

Find $\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}$ which minimizes $f(\vec{x})$

Subject to the constraints -

$$g_j(\vec{X}) \le 0 \quad , \quad j = 1, 2, \ldots m,$$

$$\text{and } h_k(\vec{X}) = 0, \quad k = m+1, m+2, \ldots n, \quad 3.1$$

where $\vec{X}$ = n dimensional vector called decision vector,

$f(\vec{X})$ = Objective function.

$g_j(\vec{X})$ = non equality constraints

$h_k(\vec{X})$ = equality constraints

If there exist no constraints in an optimization problem, it is called an unconstrained optimization problem; otherwise the problem is called constrained optimization problem.

Any Engineering or Administrative problem can be described by a set of variables. Some of these variables over which

Quadratic Programming Problems — It is a 31
NLP with a quadratic objective function and linear constraints.

L.P.P. If the objective function and all the constraints in eqn (*) are linear functions of the decision variables, the mathematical programming problem is called a Linear programming problem (LPP).

Statement:

Find $\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{Bmatrix}$

which minimizes $f(\vec{X}) = \sum_{i=1}^{n} c_i x_i$ — (**),

subject to constraints.

$$\sum_{k=1}^{n} a_{jk} x_k \le b_j, \quad j = 1, 2 \ldots m$$

and $x_i \ge 0, \quad i = 1, 2 \ldots n$.

where c_i, a_{jk} and b_j are constants.

Integer Programming problems:—

If some or all the decision variables $x_1, x_2 \ldots x_n$ of the optimization problem are constrained to take only integer values the optimization problem is called an Integer programming problem.

Classification of Optimization problems:-

- Linear Programming Problem (L.P.P)
  - Integer programming Problem (IP)
  - Non-Integer Programming problems (NIP)
- Non-linear Programming Problem (NLP Problem)
  - Integer Programming Problem. (NLIP)
  - Non-Integer programming problem. (N.L.N.I.P)
    - Geometrical Programming problem (NLGP)
    - Quadratic Programming problem. (NLQP)
    - Higher order Programming problem. (NLHOP.)

1. Linear Programming problem
2. Non-Linear Programming problems:-

If any function among the objective and constraints in equation (A) is non linear then the problem is called a non-linear programming problem. This is the most general programming problem and Geometrical programming and Quadratic programming problems are special cases of A. The techniques will be discussed latter for NLP.

3. Geometric Programming Problem:-

A geometric programming problem is one in which the objective fn and constraints are expressed as Posynomials in x_i.

the decision maker has full control are called decision variables or controlled variables and 32 others are called the parameters. The decision variables are collectively represented as a decision vector $\vec{X}$.

Applications:

To indicate the widespread scope of the subject, some typical application in different engineering discipline is given below:—

1. Design of aircraft and aerospace structure for minimum weight
2. Finding the optimal trajectories of space vehicles.
3. Design of civil Engineering structures such as frames, foundation, bridges, towers, chimneys and dams for minimum cost.
4. Design of minimum weight structures for ~~each~~ earth quake, wind and other types of random loadings.
5. Design of water resources system for obtaining maximum benefit.
6. Optimum plastic design of structures.
7. Optimum design of linkages, cams, gears, machine tools, and other mechanical devices or components.
8. Selection of machining condition in metal-cutting process for maintaining the product cost.
9. Design of material handling equipments such as conveyors, trucks and cranes, escalators for minimizing costs.
10. Design of pumps, turbines and heat transfer equipments for maximum efficiency.

- Optimal design of electrical machinery such as motors, generators and transformers.
- Optimal design of electrical networks
- Designing the shortest root to be taken by salesman to visit various cities in a single tour.
- Optimal production planning, controlling and scheduling.
- Optimum design of chemical processing equipments and plants.
- Design of optimum pipeline networks for a process industry
- Selection and layout of a site for an industry
- Planning of maintenance and replacement of equipment to reduce operating costs.

Formulation of LPP.

Example. ~~The~~ No. of workers available to operate a particular machines is limited. Or the amount of raw materials available for production process on a particular dee

These all are constraints: Formulation of LPP

Step-1. write down the decision variables of the problem

Step-2- To formulate the objective function to be optimized [Max or Min] as a linear function of decision variables.

Step-3. To formulate the other constraints/conditions of the problem such as resources, limitations, Market constraints, ~~poty~~ Interrelation between variables etc; as linear inequalities/ equations in terms of decision variables.

Definition:— Given a system of m simultaneous linear equations in $n(>m)$ unknowns, $A\bar{x} = \bar{b}$ where A is an $m\times n$ matrix and rank$(A) = m$. Let B be any $m\times m$ submatrix (non-singular) of A obtained by re-ordering m linearly independent columns of A. Then, a soln obtained by setting $n-m$ variables not associated with the columns of B equal to zero, and solving the resulting system is called a "basic solution" to the given system of equations.

The m variables, which may be all different from zero, are called "basic variables". The $m\times m$ non-singular submatrix B is called a "basis matrix" and the columns of B act as "basis vectors".

If $\bar{B}$ is the basis submatrix, then the "basic solution" to the system of equations will be $\bar{x}_B = B^{-1}\bar{b}$

Basic feasible soln:— A basic solution to the system $A\bar{x}=\bar{b}$ is called "basic feasible" if $\bar{x}_B \geq 0$.

Degenerate solution:— A basic solution to the system $A\bar{x}=\bar{b}$ is called degenerate if one or more of the basic variables vanish (or equal to zero).

Associated cost vector: Let $\bar{x}_B$ be a basic feasible soln to the LP problem,

$$\text{Maximise } Z = \bar{c}\bar{x}.$$

$$\text{Subject to } Ax = b;\ x \geq 0.$$

Then the vector $\bar{C}_B = (C_{B1}, C_{B2}, \ldots, C_{Bm})$, where C_{Bi} are components of c associated with the basic variables, is called the "cost vector" associated with the basic feasible solution $\bar{x}_B$.

Reduction of feasible solution to a basic feasible solution:—

Consider a system of m simultaneous linear eqn in $n(>m)$ variables (unknowns):—

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \bar{b}.$$

where $A = (a_1, a_2, \cdots a_n)$ is an $m\times n$ real matrix and b is a $m\times 1$ column vector of real numbers.

The (3,0). we have got — ve values at three places

Δ_1 &c. $\Delta_3 = c_3 - c_0 Y_3 = 4-(0,0,0)(0,5,4) = 4$

Suppose there exist a feasible solution to the above system of equations. Then this feasible solution can be reduced to a basic feasible solution as follows

Step-1 If the column vectors of A are linearly independent, then express one of the column vector $\bar{a}_r \in A$ as a linear combination of the remaining column vectors
$\bar{a}_r = \lambda_1 a_1 + \lambda_2 a_2 \cdots + \lambda_n a_n$ where λ_j's are scalars.
Evaluate the values of the scalars.

Step-2. Compute the ratio x_j/λ_j ; $\lambda_j > 0$; $j = 1, \cdots m$ and choose the minimum value. Let it be x_r/λ_r

Step-3. Reduce to zero the value of the variable corresponding to the minimum ratio, x_r/λ_r

Step-4. The values of new variables are given by

$$x_j' = x_j - \left(\frac{x_r}{\lambda_r}\right)\lambda_j \; ; \; j = 1, \cdots m.$$

Module-II (LPP) ①

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Q1. A firm manufactures two products A and B on which the profits earned per unit are Rs 3 and Rs 4 respectively. Each product is processed on two machines M_1 & M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2, while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hrs 30 minutes, while machine M_2 is available for 10 hrs during any working day. Find the number of units of products A and B to be manufactured to get maximum profit.

Soln. Formulation of LPP model :— Let x_1 and x_2 denote the number of units of products A and B to be produced per day.

Objective is to maximize the profit.

i.e, maximize $Z = 3x_1 + 4x_2$ —— (i). Subject to the

Constraints $x_1 + x_2 \le 450$ — (ii) (7.30 hrs = 450 Min) (10 hrs = 600 min)

$2x_1 + x_2 \le 600$ —— (iii)

$x_1 \geq 0,\ x_2 \geq 0$ —— (iv).

	A (x_1)	B (x_2)	M-Capacity
M_1	1 min	1 min.	~~7.30 hrs~~ 7.30 hrs
M_2	2 Min	1 min	10 hrs.
Profit	Rs 3	Rs 4	

Graphical Soln :— Step-1 We choose two axes as X_1 and X_2 mutually ⊥r

Step-2 The non-negativity constraints shows that the value of the variables lies only in the Ist ~~octant~~ quadrant. (i.e, in x_1-x_2-plane)

Step-3. Next we plot all the constraints/equation by changing them first into equality constraints (= sign).

x_1	x_2
0	450
450	0

x_1	x_2
0	600
300	0

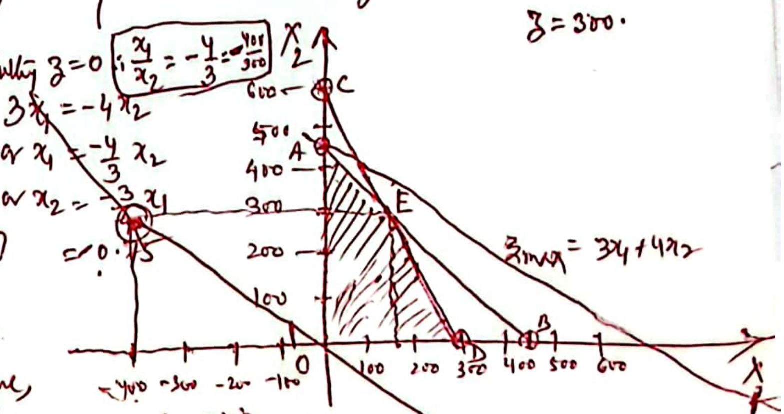
Putting $Z = 0$: $\boxed{\frac{x_1}{x_2} = -\frac{4}{3} = \frac{-400}{300}}$

$3x_1 = -4x_2$

or $x_1 = -\frac{4}{3}x_2$

or $x_2 = -\frac{3}{4}x_1$

$Z = 300$.



As the value of Z is increased from $Z = 0$, the dotted lines moving to the right, ∥ to itself. Greater the value that Z can assume, more will be the profit of Company.

We thus go on drawing the farthest way from origin these ∥ lines of Z till it passes through only one corner of the feasible region. This is the point where maxima is attained. In case of one of edge of the feasible region every point on the edge gives the max value of Z

+ve (≥0). we have got — ve values at three places

Δ_j for. $\Delta_3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$

So we have maximum of Z at $A(0, 450)$. Which means that product B should be manufactured and 450 units of this product B should proceed.

Then the daily profit will be $Z = Rs\ (0 + 4 \times 450) = Rs\ 1800/-$

This solution corresponding to point $A(0, 450)$, which maximizes the profit f_2 is called the optimal solution.

<u>Athier.</u> There are four vertices of the convex region OCDE are $O(0,0)$, $A(0, 450)$, $E(150, 300)$ and $D(300, 0)$. Putting these values in objective fns Z ~~are~~ if we get the maxm value of Z for any one of them then that will be the reqd feasible solution of ~~of~~ of LPP. which is $A(0, 450)$. Ans